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Correspondence Item

Tolerance of Non-linearities in Relay Systems*

Tolérance aux non-linéarités dans les systèmes à relais

Toleranzen von Nichtlinearitäten in Relaissystemen

Приемлемостъ нелинейностей в релейных систэмах

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Summary—It is shown that a nominally linear system with relay feedback is insensitive to unintentional time-variable gains or non-linearities in the input transducers when the relay is chattering.

THIS note considers the stability properties of relay systems having the structure indicated in Fig. 1 and the state equations

$$\dot{x}=Fx+g\beta[\mathrm{sgn}(k'x),\ t]\mathrm{sgn}(k'x) \qquad (1)$$

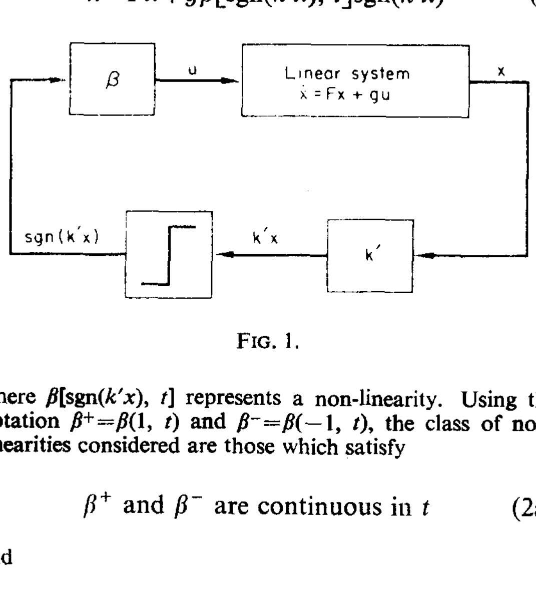

FIG. 1.

where $\beta[\mathrm{sgn}(k'x),\ t]$ represents a non-linearity. Using the notation $\beta^+=\beta(1,\ t)$ and $\beta^-=\beta(-1,\ t)$, the class of non-linearities considered are those which satisfy

$$\beta^+ \text{ and } \beta^- \text{ are continuous in } t \qquad (2a)$$

and

$$\beta^+\geq c_1>0\,; \qquad \beta^-\geq c_2>0 \qquad (2b)$$

for some positive constants c_1 and c_2. Clearly the conditions (2) will be satisfied for most non-linearities which occur in practice.

First we show, using the ideas of ANDRE and SEIBERT [1], that when the relay is in its chattering mode, the chattering motion is given by the linear state equations

$$\dot{x}=\left(F-g\frac{k'F}{k'g}\right)x\,; \qquad x(t_0)=x_0 \qquad (3)$$

where the initial state x_0 is of course in the hyperplane $k'x=0$ as are the trajectories given by Eq. (3). The significance of this result is that it is precisely the same equation of motion as occurs for the chattering mode of the relay system when the non-linearity β is replaced by an arbitrary linear gain element (see Ref. [1]). This means that the stability properties of Eq. (1) are largely independent of β, as the note goes on to show.

It is well known [2] that necessary conditions for stability near the origin are that $k'g\beta^+\leqslant 0$ and $k'g\beta^-\leqslant 0$, and that a necessary condition for chattering is that $-|k'g\beta^-|<k'Fx<|k'g\beta^+|$. From Eq. (2), it follows that the existence of a chattering regime is independent of β; note, however, that the extent of this regime does depend on β. These conditions will be assumed to hold, and in particular we first consider the special case $k'g<0$. For this case it is necessary to assume, as in [1], that associated with the relay is a small time delay.

The trajectory of the system when the relay is chattering is shown in Fig. 2. If the relay has a time delay τ, then the

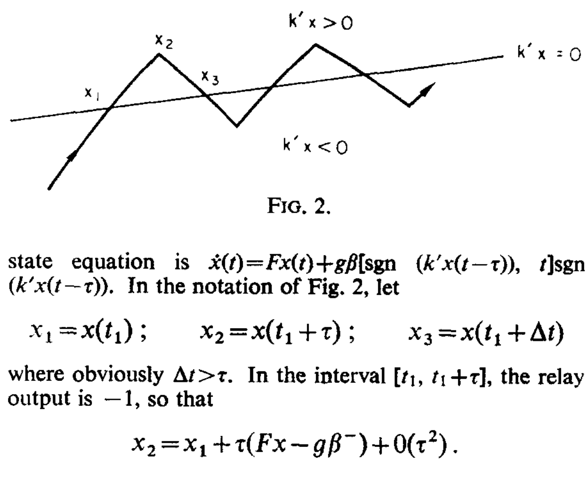

FIG. 2.

state equation is $\dot{x}(t)=Fx(t)+g\beta[\mathrm{sgn}\ (k'x(t-\tau)),\ t]\mathrm{sgn}\ (k'x(t-\tau))$. In the notation of Fig. 2, let

$$x_1=x(t_1)\,; \qquad x_2=x(t_1+\tau)\,; \qquad x_3=x(t_1+\Delta t)$$

where obviously $\Delta t>\tau$. In the interval $[t_1,\ t_1+\tau]$, the relay output is -1, so that

$$x_2=x_1+\tau(Fx-g\beta^-)+0(\tau^2)\,.$$

In the interval $[t_1+\tau,\ t_1+\Delta t]$, the control has the value $+1$, so we now have

$$\begin{aligned} x_3&=x_2+(\Delta t-\tau)(Fx+g\beta^+)+0(\tau^2)\\ &=x_1+\Delta t(Fx+g\beta^+)-\tau g(\beta^++\beta^-)+0(\tau^2)\,. \end{aligned}$$

Multiplying this equation by k' gives

$$0=\Delta t(k'Fx+k'g\beta^+)-\tau k'g(\beta^++\beta^-)+0(\tau^2)$$

or

$$\frac{\tau}{\Delta t}=\frac{k'Fx+k'g\beta^+}{k'g(\beta^++\beta^-)}+0(\tau)\,.$$

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Now from above

$$\frac{x_3 - x_1}{\Delta t} = Fx + g\beta^+ - g(\beta^+ + \beta^-)\frac{k'Fx + k'g\beta^+}{k'g(\beta^+ + \beta^-)} + 0(\tau)$$

$$= Fx - g\frac{k'F}{K'g}x + 0(\tau).$$

Clearly as $\tau \to 0$, the system trajectory approaches arbitrarily close to the trajectory defined by Eq. (3). This means that if the feedback gain k is chosen so that Eq. (3) is asymptotically stable on the hyperplane $k'x=0$, then the original system (1) is asymptotically stable in its chattering mode for arbitrary β satisfying Eq. (2). The theory of relay systems, see for example [2], now gives the result that since system (1) is asymptotically stable in its chattering mode it is also asymptotically stable for *all* sufficiently small initial states.

We conclude that in order to ensure (at least) local asymptotic stability and chattering mode asymptotic stability the feedback gain k may be chosen independently of the non-linearity β.

Consider now the case when $k'g=0$. From a result of Anosov [3], it appears that a necessary condition for stability in this case is $k'Fg<0$, so we consider only this condition. The result is then as follows: there is no "chattering", in the sense of Ref. [1], but all trajectories suitably close to the origin eventually approach arbitrarily close to the hyperplane $k'x=k'Fx=0$, and the effective trajectory is then given by

$$\dot{x} = \left(F - g\frac{k'F^2}{k'Fg}\right)x. \qquad (4)$$

The result is derived as follows. The theory of [1] is sufficient to show that in this case the state trajectory near the switching line is as in Fig. 3, i.e. it spirals around the intersection of the two hyperplanes $k'x=0$ and $k'Fx=0$.

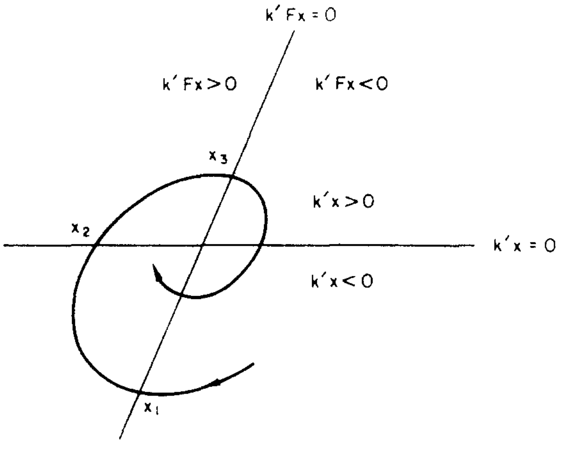

Fig. 3.

Also, the switching is *regular* [1] and so it is unnecessary to assume any time delay. In the notation of Fig. 3, then,

$$x_2 = x_1 + \tau_{12}(Fx - g\beta^-) + 0(\tau_{13}^{\ 2})$$

and

$$x_3 = x_2 + (\tau_{13} - \tau_{12})(Fx + g\beta^+) + 0(\tau_{13}^{\ 2})$$

where τ_{12} and τ_{13} are the times taken to go from x_1 to x_2 and x_3 respectively. These equations may be solved, as before, to give

$$\frac{x_3 - x_1}{\tau_{13}} = \left(F - g\frac{k'F^2}{k'Fg}\right)x + 0(\tau_{13})$$

whence the result is obvious.

Moreover, from Ref. [1] or [2], local asymptotic stability follows if chattering occurs (i.e. $k'Fg\beta<0$) and if the chattering trajectories are asymptotically stable. These conditions are independent of β, provided only that Eq.(2) holds. It should be noted, however, that the region of local asymptotic stability does depend on β.

In the design of relay feedback systems that are at least locally asymptotically stable, the choice of k is governed by the requirement that $k'g<0$ (or $k'Fg<0$) and that the coefficient matrices of Eqs. (3) or (4) have $(n-1)$ or $(n-2)$ eigenvalues with negative real parts. (It is easily shown, see Ref. [2], that the other eigenvalues are zero.) The significance of the present theory is that it shows that a controller design can be carried out with little or no knowledge of any unintentional non-linearities at the system input. The results also hold for a large class of input disturbances, since these may in many cases be included in the non-linearity β.

References

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Résumé—Il est montré qu'un système nominalement linéaire avec reaction à relais est insensible aux variations accidentelles des gains dans le temps ou aux non-linéarités dans les capteurs d'entrée lorsque le relais vibre.

Zusammenfassung—Gezeigt wird, daß ein nominell lineares System mit Relais-Rückführung unempfindlich ist gegenüber unabsichtlicher zeitvariabler Verstärkung oder Nichtlinearitäten in den Eingangswandlern, wenn das Relais prellt.

Резюме—Показывается что номинально-линейная система с релейной обратной связью нечувствительна к случайным изменениям по времени коэффициентов усиления или к нелинейностям в входных датчиках когда реле вибрирует.